Sum of Independent Poisson Distributions

If two independent random variables X and Y both follow a Poisson distribution, they can be combined into one variable. The mean of the new variable will be a sum of the mean of X and Y.

$$X \sim Po(\lambda_1),\ Y \sim Po(\lambda_2) \text{ and } Z = X + Y$$

$$\Rightarrow Z \sim Po(\lambda_1 + \lambda_2)$$

Example 1: The number of phone calls a company receives in a day is thought to follow a Poisson distribution with a mean of 5. If the number of emails the company receives in a day also follows a Poisson distribution, but with a mean of 12, find the probability that the total number of phone calls and emails received in a day by the company exceeds 16, stating any assumptions made in your calculations.

Let X be the number of phone calls received by the company in a day and state its distribution.	$X \sim Po(5)$
Let Y be the number of emails received by the company in a day and state its distribution.	$Y \sim Po(12)$
Let Z be the total number of phone calls and emails received by the company in a day and state its distribution.	$Z = X + Y$ $Z \sim Po(12 + 5)$ $\Rightarrow Z \sim Po(17)$
Find $P(Z > 16)$. Relate the probability to $P(Z \le k)$ to use the calculator function.	$P(Z > 16) = 1 - P(Z \le 16)$ $= 1 - 0.46774$ $= 0.532$ (3 s.f.)
Identify that Poisson distributions can only be summed if they are independent.	The number of phone calls and the number of emails received by the company are independent of each other.

Scaling a Poisson Distribution

Similarly, Poisson distributions can be scaled. For a random variable X with a Poisson distribution, the mean of the new Poisson distribution will be the original mean multiplied by the same scale factor as the scale factor for the random variable.

$$X \sim Po(\lambda), Y = aX$$

$$Y \sim Po(a\lambda)$$

where a is a constant.

Example 2: Given that the number of customers entering a shop follows a Poisson distribution and the average number of customers entering the shop is 13 per hour, find the probability of having more than 92 customers over an 8 hour period when the shop is open.

Let X be the number of customers entering the shop in an hour.	$X \sim Po(13)$
Let Y be the number of customers entering the shop in 8 hours.	$Y = 8X$ $Y \sim Po(13 \times 8)$ $\Rightarrow Y \sim Po(104)$
Find $P(Y > 92)$.	$P(Y > 92) = 1 - P(Y \le 92)$ $= 1 - 0.12865$ $= 0.871$ (3 s.f.)

Hypothesis Testing Using Poisson Distributions

Hypothesis testing can be used to test if the mean has changed for a variable following a Poisson distribution, or if a set of observations belongs to a given Poisson distribution. The null hypothesis (H_0) usually states that the mean has not been changed, whereas the alternative hypothesis (H_1) states that there is a change in mean.

The probability of obtaining the observed outcome, or an outcome which is more extreme, is then calculated. If this is less than the significance level, it provides evidence supporting the H_1, and H_0 can be rejected. Otherwise, H_0 is accepted. If H_1 states that the mean has changed, the hypothesis test looks at a change in either direction. This is a two-tailed test and the significance level at each tail should be half of the total significance level. If H_1 specifically states that the mean has either increased or decreased, this is a one-tailed test, and the significance level does not need to be halved at each tail.

Example 3: A factory manufactures product X. On average, there are 7 faulty products out of every 100 products for product X. A new machine has been bought and out of a 500 products, only 15 were faulty. Test at the 5% significance level if the new machine has decreased the rate of error.

Let X be the number of faulty products out of every 100 products.	$X \sim Po(7)$
State the null and alternative hypothesis.	$H_0: \lambda = 7$ $H_1: \lambda < 7$
Find the probability of obtaining 15 or less faulty products out 500 products. Note that this observation is for 500 products, whereas the average of 7 is for 100 products. The observation of 15 faulty products needs to be changed to variable X first.	$\frac{15}{5} = 3$ faulty products in every 100 products $P(X \le 3) = 0.0818$ (3 s.f.)
Compare the probability with the significance value given (0.05). Since this is a one-tailed test, there is no need to split the significance value into two tails.	$0.0818 > 0.05$
State your conclusion, giving a reason.	Accept H_0 as $P(X \le 3)$ is more than 5%. There is insufficient evidence to show that the rate of error has decreased from 7 per 100 products.

Example 4: A random variable X follows the Poisson distribution $X \sim Po(9)$. Find the critical region for the null hypothesis to be rejected for $\lambda \ne 9$. State whether an observation of 14 falls under this distribution.

State the null and alternative hypotheses.	$H_0: \lambda = 9$ $H_1: \lambda \ne 9$
Find the value of a such that $P(X \le a) < 0.025$, and the value of b such that $P(X \ge b) < 0.025$. Note that this is a two-tailed test so the significance level needs to be halved at each end.	$P(X \le 3) = 0.02123$ $P(X \le 4) = 0.05496$ $a = 3$ $\therefore X \le 3$ is the critical region at the lower tail. $P(X \ge b) < 0.025$ $1 - P(X < b) < 0.025$ $P(X < b) > 0.975$ $P(X < 16) = 0.97796$ $P(X < 15) = 0.95853$ $b = 16$ $\therefore X \ge 16$ is the critical region at the upper tail.
Check whether the observation falls under the critical region.	14 does not fall into either of the critical regions, so the null hypothesis is accepted. The observation falls under the Poisson distribution.